Some Monotonicity and Convexity Results for Integral Means

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Analogues and extensions of the classical result on monotonicity of L^p norms and log convexity of p^{th} power of L^p norms are established.

In the discussion of criteria for magnetohydrodynamic stability one encounters inequalities of the type

$$\{\oint B^{-1}\,\mathrm{d}l\}\oint B^{-3}\,\mathrm{d}l \geq \{\oint B^{-2}\,\mathrm{d}l\}^2\,, \qquad (1\,\mathrm{a})$$

$$\{\oint B^{-1}\,\mathrm{d}l\}\oint B^{-4}\,\mathrm{d}l \geq \{\oint B^{-2}\,\mathrm{d}l\}\oint B^{-3}\,\mathrm{d}l\,, \qquad (1\,\mathrm{b})$$

where l is the arc length, $B(l)$ the (non-vanishing) magnetic field, and $\oint \ldots \mathrm{d}l$ the integral over a closed line of force. While (1 a) is of the Schwarz type, (1 b) is not. Inequalities (1) can be reduced to monotonicity of the functional

$$\left\{\int_0^1 f^\alpha(x)\,\mathrm{d}\mu\right\}\int_0^1 f^{-\alpha}(x)\,\mathrm{d}\mu, \quad 0<f<\infty$$

in the real parameter α with a positive weight function $\mu(x)$.

The aim here is to establish certain integral inequalities which are in the spirit of the following classical result on L^p norms (see, for instance, [1]):

Given a measure space $(\Omega, \mathscr{A}, \mu)$, where μ is a probability measure, and given f a non-negative real-valued function on Ω, the functional $I(p) = \int_\Omega f^p\,\mathrm{d}\mu$, defined for $p \in R$, is log convex in p.

The log convexity result in Theorem 4 (see below) is a direct extension of [1] to the case where, instead of one, several independent functions are considered.

Here is a brief description of the other results of the present paper: For $\alpha=(\alpha_1, \ldots, \alpha_m)$ a multi-index, let $I_A(\alpha)$ be defined by

$$I_A(\alpha_1, \ldots, \alpha_m) = \left\{\int_A \prod_{i=1}^m f_i^{\alpha_i}\,\mathrm{d}\mu\right\}\left\{\int_A \prod_{i=1}^m f_i^{-\alpha_i}\,\mathrm{d}\mu\right\}. \quad (2)$$

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Theorems 1 and 2 show that if μ is a positive measure (not necessarily of total mass one), then $I_A(\alpha)$ is a monotonic function of $|\alpha_i|$ for each $i=1, \ldots, m$ provided that f satisfies Condition (δ) below, which also turns out (see Theorem 2) to be necessary in order for the monotonicity to hold for all A. In Theorem 3 we show that I_A is convex and that, if a slightly more restrictive (though simpler) condition than (δ) is imposed, i.e.

$$\text{Condition } (\gamma)\text{:} \quad \begin{cases} f_i(x) \geq f_i(y) \;\Rightarrow\; f_j(x) \geq f_j(y), & j \neq i, \\ i,j=1, \ldots, m, & \forall (x,y) \in A \times A \end{cases},$$

then all m^2 second partial derivatives of I_A are non-negative.

Theorem 1. *Let $(\Omega, \mathscr{A}, \mu)$ be a positive measure space and let $f\colon \Omega \to R_+$ be a measurable function. Then the functional $I_A(\alpha)$ for $\alpha \in R$ and $A \subset \Omega$ (see (2)) is monotonically increasing in $|\alpha|$ unless f is constant.*

First Proof: It suffices to consider the case $\alpha \geq 0$ since $I_A(-\alpha) = I_A(\alpha)$. Let us begin by observing that in order for $\int_A f^\alpha\,\mathrm{d}\mu$ and $\int_A f^{-\alpha}\,\mathrm{d}\mu$ both to be finite, we must have $\mu(A) < \infty$. To see this, note that

$$\mu(A) = \mu\{x\colon f(x) \geq 1\} + \mu\left\{x\colon \frac{1}{f(x)} > 1\right\}$$

$$\leq \int_A f^\alpha\,\mathrm{d}\mu + \int_A f^{-\alpha}\,\mathrm{d}\mu\,.$$

By rescaling we may assume, without loss of generality, that $\mu(A)=1$. If $\alpha < \beta$, we have, by applying Jensen's inequality $g(\int h\,\mathrm{d}\mu) \leq \int g(h)\,\mathrm{d}\mu$ with the con-

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vex function $g(u) = u^{\beta/\alpha}$,

$$(I_A(\alpha))^{\beta/\alpha} \le \Big(\int_A f^\beta \,\mathrm{d}\mu\Big)\Big(\int_A f^{-\beta} \,\mathrm{d}\mu\Big) = I_A(\beta)\,.$$

It thus follows that

$$I_A(\alpha) \le I_A(\beta)^{\alpha/\beta}\,.$$

It suffices to prove $I_A(\beta) > 1$ for every β (unless $f \equiv$ constant). But this is easy for we have, by the Cauchy-Schwarz inequality,

$$1 = \int_A f^{\beta/2} f^{-\beta/2} \,\mathrm{d}\mu \le \Big(\int_A f^\beta \,\mathrm{d}\mu\Big)^{1/2}\Big(\int_A f^{-\beta} \,\mathrm{d}\mu\Big)^{1/2} = \sqrt{I_A(\beta)}\,,$$

with equality only if $f^{\beta/2}$ is a multiple of $f^{-\beta/2}$, i.e. only if $f = c$.

Example: Taking μ as counting measure, we have for $\alpha_i > 0$, $i = 1, \ldots, N$ as an application of Theorem 1 the following result for sums:

$$\Big(\sum_{i=1}^N a_i^\alpha\Big)\Big(\sum_{i=1}^N a_i^{-\alpha}\Big)$$

is a monotonically increasing function of $|\alpha|$.

Second proof: Let us write $I_A(\alpha)$ as

$$I_A(\alpha) = \frac{1}{2}\Big[\Big(\int_A f^\alpha(x)\,\mathrm{d}\mu\,(x)\Big)\Big(\int_A f^{-\alpha}(y)\,\mathrm{d}\mu(y)\Big) + \Big(\int_A f^\alpha(y)\,\mathrm{d}\mu(y)\Big)\Big(\int_A f^{-\alpha}(x)\,\mathrm{d}\mu(x)\Big)\Big]$$

$$= \frac{1}{2}\int_A\int_A \Big[\Big(\frac{f(x)}{f(y)}\Big)^\alpha + \Big(\frac{f(y)}{f(x)}\Big)^\alpha\Big]\,\mathrm{d}\mu(x)\,\mathrm{d}\mu(y)$$

and use the fact that

$$\alpha \to a^\alpha + \frac{1}{a^\alpha}\,, \quad \text{for} \quad a > 0\,,$$

is a monotonically increasing function of $|\alpha|$ (unless obviously $a = 1$).

This second proof has the advantage of providing the following m-variable extension.

Theorem 2. *The $(\Omega, \mathscr{A}, \mu)$ be a positive measure space and $f_1, \ldots, f_m\colon \Omega \to R^+$ be measurable functions, then $I_A(\alpha_1, \ldots, \alpha_m)$ increases on each variable separately for every A in $\mathscr{A}$ if, and only if, Condition (δ) holds, i.e.*

$$\text{Condition}(\delta)\colon \begin{cases} \alpha_i \dfrac{\partial}{\partial \alpha_i} F_{x,y} \ge 0 & \textit{for each variable } \alpha_i \textit{ separately,} \\ \forall\,(x, y) \in \Omega \times \Omega \textit{ a.e. } \mathrm{d}\mu(x) \otimes \mathrm{d}\mu(\mu). \end{cases}$$

where

$$F_{x,y}(\alpha_1, \ldots, \alpha_m) = \prod_{i=1}^m \Big(\frac{f_i(x)}{f_i(y)}\Big)^{\alpha_i} + \prod_{i=1}^m \Big(\frac{f_i(y)}{f_i(x)}\Big)^{\alpha_i}\,.$$

Remark. Note that Condition (γ) implies Condition (δ). Indeed, setting

$$a^\alpha_{x,y}(\alpha_1, \ldots, \alpha_m) = \prod_{i=1}^m \Big(\frac{f_i(x)}{f_i(y)}\Big)^{\alpha_i}$$

and $a_i = f_i(x)/f_i(y)$, we have

$$\frac{\partial F}{\partial \alpha_i} = (a^\alpha - a^{-\alpha}) \log a_i$$

for $i = 1, \ldots, m$. But using the definition of a_i and Condition (γ) we see that

$$\log a_i \ge 0 \quad\Leftrightarrow\quad \frac{f_i(x)}{f_i(y)} \ge 1 \quad\Leftrightarrow\quad \frac{f_j(x)}{f_j(y)} \ge 1\,,$$

$$\forall j \ne i,\ j = 1, \ldots, m \ \Rightarrow\ a^\alpha - a^{-\alpha} \ge 0\,,$$

and similarly

$$\log a_i \le 0 \ \Rightarrow\ a^\alpha - a^{-\alpha} \le 0\,.$$

If Condition (γ) holds, we can in fact say more, i.e. we have

Theorem 3. *Let $(\Omega, \mathscr{A}, \mu)$ be a positive measure space and $f_1, \ldots, f_m\colon \Omega \to R^+$ be measurable functions, then $I_A(\alpha_1, \ldots, \alpha_m)$ is a convex function of $(\alpha_1, \ldots, \alpha_m)$. If, in addition, Condition (γ) is satisfied, then all m^2 second partial derivatives of I_A with respect to α_i are non-negative.*

Proof: We begin by calculating the second-order partials

$$\frac{\partial^2 I}{\partial \alpha_i\,\partial \alpha_j} = \int_\Omega (a^\alpha + a^{-\alpha}) \log a_i \log a_j \,\mathrm{d}\mu\,.$$

But

$$\text{Condition } \gamma \ \Rightarrow\ (\log a_i)(\log a_j) \ge 0\,.$$

We next establish the convexity. If we let

$$c_i = \log a_i\,, \quad i = 1, \ldots, m\,,$$

the positive semidefiniteness of the matrix $\partial^2 I/\partial \alpha_i\,\partial \alpha_j$ is an immediate consequence of the positive semidefiniteness of the matrix $D = d_{ij}$, where $d_{ij} = c_i c_j$, and of the positivity of the factor $a^\alpha + a^{-\alpha}$.

That D is positive definite is, in turn, elementary:

$$d_{ij} \xi_i \xi_j = c_i \xi_i c_j \xi_j \geq 0 .$$

Proof of Theorem 2: The sufficiency is immediate for we can write

$$I_A(\alpha_1, \ldots, \alpha_m) = \iint_{\mathscr{A} \times \mathscr{A}} F_{x,y}(\alpha_1, \ldots, \alpha_m) \, d\mu(x) \, d\mu(y) .$$

To see the necessity, let us assume that $\exists \alpha_1 \leq \bar{\alpha}_1, \ldots, \alpha_m \leq \bar{\alpha}_m$ so that

$$F_{x,y}(\alpha_1, \ldots, \alpha_m) > F_{x,y}(\bar{\alpha}_1, \ldots, \bar{\alpha}_m)$$

for (x, y) in a set $B \in \Omega \times \Omega$ of positive $d\mu(x) \times d\mu(y)$ measure. Since $F_{x,y} = F_{y,x}$ we see that

$$(x, y) \in B \iff (y, x) \in B ,$$

so that B must be of the form $A \times A$ for $A \in \mathscr{A}$ with $\mu(A) > 0$. In particular,

$$I_A(\alpha_1, \ldots, \alpha_m) > I_A(\bar{\alpha}_1, \ldots, \bar{\alpha}_m) .$$

Another generalization of Hardy, Littlewood, and Polya's result is given by the following theorem.

Theorem 4. *Let $(\Omega, \mathscr{A}, \mu)$ be a positive measure space with μ a probability measure. Let $f_i \colon \Omega \to R^+$ be measurable functions. Let $(\alpha_1, \ldots, \alpha_m) \in \{(-\infty, 0)^m\} \cup \{(0, \infty)^m\}$, then*

$$J(\alpha_1, \ldots, \alpha_m) = \int_\Omega f_1^{\alpha_1}(x) \ldots f_m^{\alpha_m}(x) \, d\mu(x)$$

is log convex as a function of $\alpha = (\alpha_1, \ldots, \alpha_m)$. Moreover, if, in addition, we assume that Condition (γ) holds, then all m^2 second partials of $\log J$ *are nonnegative.*

Proof: Let

$$b(\alpha_1, \ldots, \alpha_m) = \prod_{i=1}^{m} f_i^{\alpha_i}(x) .$$

Consider $K(\alpha_1, \ldots, \alpha_m) = \log J(\alpha_1, \ldots, \alpha_m)$. We have

$$\frac{\partial K}{\partial \alpha_i} = \int_\Omega b \log f_i \, d\mu \left\{ \int_\Omega b \, d\mu \right\}^{-1}$$

and

$$\frac{\partial^2 K}{\partial \alpha_i \, \partial \alpha_j} = \left\{ \left[\int_\Omega b(x) \log f_i(x) \log f_j(x) \, d\mu(x) \right] \left[\int_\Omega b(y) \, d\mu(y) \right] - \left[\int_\Omega b(x) \log f_i(x) \, d\mu(x) \right] \cdot \left[\int_\Omega b(y) \log f_j(y) \, d\mu(y) \right] \right\} \left\{ \int_\Omega b \, d\mu \right\}^{-2} .$$

Interchanging the role of x and y and adding, we find that the numerator on the right-hand side may be expressed as

$$\tfrac{1}{2} \iint_{\Omega \times \Omega} [\log f_i(x) - \log f_i(y)] \cdot [\log f_j(x) - \log f_j(y)] \, dv(x) \, dv(y) ,$$

where we have set

$$dv = b \, d\mu \quad (\text{recall } b > 0) .$$

Condition (γ) makes the integrand clearly nonnegative, thus $\partial^2 K / \partial \alpha_i \, \partial \alpha_j \geq 0$. Setting

$$\log f_i(x) - \log f_i(y) = c_i ,$$

we see, as in the proof of Theorem 3, that the positive semidefiniteness of the matrix $\partial^2 K / \partial \alpha_i \, \partial \alpha_j$ follows immediately from that of the matrix $D = \{d_{ij}\} = \{c_i c_j\}$.

[1] G. H. Hardy, J. E. Littlewood, and G. Polya, Inequalities. Cambridge University Press, Cambridge 1967, p. 145.